Exercises on Laplace Transformation

(1) Find the Laplace Transform of each of the following functions:

(a) $f(t) = e^{-2t}\cos^2 3t - 3t^2 e^{3t}$; (b) $f(t) = e^t \cdot \dfrac{d^{100}}{dt^{100}}(e^{-t}\,t^{100})$.

(2) Compute the inverse Laplace Transform of

(a) $F(s) = \dfrac{4s+5}{(s-2)^3(s+3)}$; (b) $F(s) = \dfrac{1}{s^2(s^2+3s-4)}$;

(c) $F(s) = \dfrac{s}{(s-1)^2(s^2+2s+5)}$.

(3) Solve the following initial valued problems by Laplace Transform

(a) $y'' + k^2 y = 2\sin\omega t$, with $y(0) = y'(0) = 0$;

(b) $y'' + 4y' + 5y = 100\,e^{-2t}$, with $y(0) = -1$ and $y'(0) = 0$;

,

Solutions to Laplace Transform Exercise

(1) (a) Since $\cos^2 3t = \dfrac{1+\cos 6t}{2}$, $\mathfrak{L}\{e^{-2t}\cos^2 3t - 3t^2e^{3t}\}$

$$= \mathfrak{L}\left\{e^{-2t}\left(\tfrac{1}{2}+\tfrac{1}{2}\cos 6t\right) - 3e^{3t}t^2\right\} = \frac{1}{2(s+2)} + \frac{1}{2}\frac{s+2}{(s+2)^2+36} - \frac{6}{(s-3)^3}$$

$$= \frac{1}{2(s+2)} + \frac{1}{2}\frac{s+2}{s^2+4s+40} - \frac{6}{(s-3)^3}.$$

(b) $\because \mathfrak{L}\{e^{-t}t^{100}\} = \dfrac{100!}{(s+1)^{101}}$, $\therefore \mathfrak{L}\left\{\dfrac{d^{100}}{dt^{100}}(e^{-t}t^{100})\right\} = s^{100}\times\dfrac{100!}{(s+1)^{101}}$

because $\dfrac{d^k}{dt^k}(e^{-t}t^{100}) = 0$ when $t=0$ whenever $0 \le k \le 99$. Therefore,

$$\mathfrak{L}\left\{e^t\cdot\frac{d^{100}}{dt^{100}}(e^{-t}t^{100})\right\} = 100!\times\frac{(s-1)^{100}}{s^{101}}.$$

(2) (a) Decomposing $F(s)$ into partial fractions, we obtain

$$\frac{4s+5}{(s-2)^3(s+3)} = \frac{7}{125\,(s+3)} - \frac{7}{125\,(s-2)} + \frac{7}{25\,(s-2)^2} + \frac{13}{5\,(s-2)^3}.$$

Therefore, $\mathfrak{L}^{-1}\{F(s)\} = \dfrac{7}{125}e^{-3t} - \dfrac{7}{125}e^{2t} + \dfrac{7}{25}te^{2t} + \dfrac{13}{10}t^2e^{2t}$.

(b) $\mathfrak{L}^{-1}\left\{\dfrac{1}{s^2(s^2+3s-4)}\right\} = \mathfrak{L}^{-1}\left\{\dfrac{1}{5\,(s-1)} - \dfrac{3}{16s} - \dfrac{1}{80\,(s+4)} - \dfrac{1}{4s^2}\right\}$

$$= \frac{1}{5}e^t - \frac{3}{16} - \frac{1}{80}e^{-4t} - \frac{1}{4}t.$$

(c) We decompose $F(s)$ into partial fractions to obtain

$$\frac{s}{(s-1)^2(s^2+2s+5)} = \frac{1}{16\,(s-1)} + \frac{1}{8\,(s-1)^2} - \frac{1}{16}\frac{s}{s^2+2s+5} - \frac{5}{16}\frac{1}{s^2+2s+5}.$$

Since $s^2+2s+5 = (s+1)^2+2^2$, we conclude that

$$\mathfrak{L}^{-1}\{F(s)\} = \frac{e^t}{16} + \frac{te^t}{8} - \frac{e^{-t}}{16}(\cos 2t - \frac{\sin 2t}{2}) - \frac{5}{32}e^{-t}\sin 2t.$$

(3) (a) We transform the differential equation to obtain

$$(s^2+k^2)Y(s) = \frac{2\omega}{s^2+\omega^2},$$

from which it follows that $Y(s) = \dfrac{2\omega}{(s^2+\omega^2)(s^2+k^2)}$. There are *two cases* to consider:

(i) If $\omega = k$, then $y(t) = \mathfrak{L}^{-1}\left\{\frac{2\omega}{(s^2+\omega^2)^2}\right\} = \frac{\sin\omega t - \omega t\cos\omega t}{\omega^2}$.

(ii) If $\omega \neq k$, then $y(t) = \frac{2\omega}{k^2-\omega^2}\left(\frac{\sin\omega t}{\omega} - \frac{\sin kt}{k}\right)$.

(b) We apply Laplace Transform to the equation to obtain

$$(s^2+4s+5)Y(s) + s + 4 = \frac{100}{s+2}.$$

Hence $Y(s) = -\frac{s}{s^2+4s+5} - \frac{4}{s^2+4s+5} + \frac{100}{(s+2)(s^2+4s+5)}$. Using the fact that $\frac{100}{(s+2)(s^2+4s+5)} = \frac{100}{s+2} - \frac{100s}{s^2+4s+5} - \frac{200}{s^2+4s+5}$, one obtains

$$\begin{aligned} y(t) &= \mathfrak{L}^{-1}\left\{-\frac{s}{s^2+4s+5} - \frac{4}{s^2+4s+5} + \frac{100}{(s+2)(s^2+4s+5)}\right\} \\ &= \mathfrak{L}^{-1}\left\{-\frac{s}{(s+2)^2+1} - \frac{4}{(s+2)^2+1} + \frac{100}{(s+2)[(s+2)^2+1]}\right\} \\ &= -100e^{-2t} - 101e^{-2t}(\cos t - 2\sin t) - 204e^{-2t}\sin t. \end{aligned}$$